Vector Spaces and Linear Transformations

Vector spaces A vector space is a nonempty set V , whose objects are called vectors, equipped with two operations, called addition and scalar multiplication: For any two vectors u, v in V and a scalar c, there are unique vectors u + v and cu in V such that the following properties are satisfied.

1. u + v = v + u,

2. (u + v) + w = u + (v + w),

3. There is a vector 0, called the zero vector, such that u + 0 = u,

4. For any vector u there is a vector −u such that u + (−u) = 0;

5. c(u + v) = cu + cv,

6. (c + d)u = cu + du,

7. c(du) = (cd)u,

8. 1u = u.

By definition of vector space it is easy to see that for any vector u and scalar c,

0u = 0, c0 = 0, −u = (−1)u.

For instance,

Example1.1.

(a) The Euclidean space R^n is a vector space under the ordinary addition and scalar multiplication.

(b) The set P_n of all polynomials of degree less than or equal to n is a vector space under the ordinary addition and scalar multiplication of polynomials.

(c) The set M(m, n) of all m × n matrices is a vector space under the ordinary addition and scalar multiplication of matrices.

(d) The set C[a, b] of all continuous functions on the closed interval [a, b] is a vector space under the ordinary addition and scalar multiplication of functions.

Definition 1.1. Let V and W be vector spaces, and W ⊆ V. If the addition and scalar multiplication in W are the same as the addition and scalar multiplication in V, then W is called a subspace of V .

If H is a subspace of V, then H is closed for the addition and scalar multiplication of V, i.e., for any u, v ∈ H and scalar c ∈ R,

we have u + v ∈ H, cv ∈ H.

For a nonempty set S of a vector space V, to verify whether S is a subspace of V , it is required to check (1) whether the addition and scalar multiplication are well defined in the given subset S, that is, whether they are closed under the addition and scalar multiplication of V ; (2) whether the eight properties (1-8) are satisfied. However, the following theorem shows that we only need to check (1), that is, to check whether the addition and scalar multiplication are closed in the given subset S.

SUBSPACES

Let V be a vector space over a field F. A non-void subset S of V is said to be a subspace of V if S itself is a vector space over F under the operations on V restricted to S.

If V is a vector space over a field F, then the null (zero) space $\{0_V\}$ and the entire space V are subspaces of V. These two subspaces are called trivial (improper) subspaces of V and any other subspace of V is called a non-trivial (proper) subspace of V.

Q1. Show that $S = \{(0, b, c) : b, c \in R\}$ is a subspace of real vector space R^3.

SOLUTION: Obviously, S is a non-void subset of R^3. Let $u = (0, b_1, c_1)$, $v = (0, b_2, c_2) \in S$ and $\lambda, \mu \in R$. Then,

$\lambda u+\mu v = \lambda(0, b_1, c_1) +\mu(0, b_2, c_2)$

$\Rightarrow \lambda u+\mu v = (0, \lambda b_1 +\mu b_2, \lambda c_1 +\mu c_2) \in S$.

Hence, S is a subspace of R^3.

Linear independence and linear dependence

Let $S = \{v_1,\dots ,v_k\} \subset V$, a vector space. We say that S is linearly dependent (l.d.) if there are scalars $a_1,\dots ,a_k$ not all zero for which

$a_1v_1 + a_2v_2 + \cdots + a_kvk = 0$.

Otherwise we say S is linearly independent (l.i.).

Proposition : If $S = \{v_1,\dots ,v_k\} \subset V$, a vector space, is linearly dependent, then one member of this set can be expressed as a linear combination of the others.

Proof. We know that there are scalars $a_1,\dots ,a_k$

such that $a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0$

Since not all of the coefficients are zero, we can solve for one of the vectors as a linear combination of the other vectors.

Q1. In the vector space of polynomials P3, determine if the set S is linearly independent or linearly dependent. S = $2 + x - 3x_2 - 8x_3$, $1 + x + x_2 + 5x_3$, $3 - 4x_2 - 7x_3$

Massage according to the definitions of scalar multiplication and vector addition in the definition of P3 (Example VSP) and use the zero vector for this vector space

$(2a_1 + a_2 + 3a_3) + (a_1 + a_2) x + (-3a_1 + a_2 - 4a_3) x^2 + (-8a_1 + 5a_2 - 7a_3) x^3 = 0 + 0x + 0x^2 + 0x$

The definition of the equality of polynomials allows us to deduce the following four equations

$2a_1 + a_2 + 3a_3 = 0$

$a_1 + a_2 = 0$

$-3a_1 + a_2 - 4a_3 = 0$

$-8a_1 + 5a_2 - 7a_3$

0Row-reducing the coefficient matrix of this homogeneous system leads to the unique solution $a_1 = a_2 = a_3 = 0$. So the only relation of linear dependence on S is the trivial one, and this is linear independence for S (Definition LI)

Q2 Express (1, 7, -4) as a linear combination of the vectors (1, -3, 2) and (2,-1, 1) in the vector space V_3 of real numbers R.

Solution. Let (1, 7, -4) = a(1, -3, 2) + b(2,-1, 1)

= (a+2b, -3a-2b, 2a+b)

Then, we have

a+2b=1

-3a-b=7

2a+b=-4

Solving the first and second equation, we get a= -3 and b=2. The values of a=-3 and b=2 also satisfy the third equation.

Thus (1, 7, -4) = -3(1, -3, 2) + 2(2,-1, 1) is the required linear combination.

Bases:

The idea of a basis is that of finding a minimal generating set for a vector space.

Let V be a vector space and S = $\{v_1,\dots ,v_k\} \subset V$. We call S a spanning set for the subspace U = S(S).

Suppose that V is a vector space, and S = $\{v_1,\dots ,v_k\}$ is a linearly independent spanning set for V . Then S is called a basis of V. Modify this definition correspondingly for subspaces.

or

Let H be a subspace of a vector space V . An ordered set B = $\{v_1, v_2, \ldots, v_p\}$ of vectors in V is called a basis for H if

(a) B is a linearly independent set, and
(b) B spans H, that is, H = Span $\{v_1, v_2, \ldots, v_p\}$.

Dimensions of vector spaces:

A vector space V is said to be finite dimensional if it can be spanned by a set of finite number of vectors. The dimension of V, denoted by dim V, is the number of vectors of a basis of V . The dimension of the zero vector space {0} is zero. If V cannot be spanned by any finite set of vectors, then V is said to be infinite dimensional.

Or

A vector space V(F) is said to be a finite dimensional vector space if there exists a finite subset of V that spans it. A vector space which is not finite dimensional may be called an infinite dimensional vector space.

Q1. Show that the set B = $\{1, x, x^2,\ldots,x^n\}$ is a basis for the vector space of all real polynomials of degree not exceeding n.

SOLUTION : Let $\lambda_0, \lambda_1,\ldots,\lambda_n \in R$ be such that

$\lambda_0 \cdot 1+\lambda_1 \cdot x+\cdots+\lambda_n x^n = 0(x)$ (zero polynomial)

Then, $\lambda_0 \cdot 1+\lambda_1 \cdot x+\cdots+\lambda_n x^n = 0(x)$ (zero polynomial)

$\Rightarrow \lambda_0 +\lambda_1 x+\lambda_2 x^2 +\cdots+\lambda_n x^n = 0+0x+0x^2 +\cdots+0x^n +\ldots$

$\Rightarrow \lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$

Therefore, the set B is linearly independent. Also, the set B spans the vector space $P_n(x)$ of all real polynomials of degree not exceeding n, because every polynomial of degree less than or equal n is a linear combination of B. Hence, B is a basis for the vector space $P_n(x)$ of all real polynomials of degree not exceeding n.

Q2. Find the coordinates of the vector (a,b,c) in the real vector space R^3 relative to the ordered basis (b_1,b_2,b_3), where $b_1 = (1,0,-1)$, $b_2 = (1,1,1)$, $b_3 = (1,0,0)$.

SOLUTION: Let $\lambda,\mu, \gamma \in R$ be such that

$(a,b,c) = \lambda(1,0,-1) +\mu(1,1,1) +\gamma(1,0,0)$.

$\Rightarrow (a,b,c)=(\lambda+\mu+\gamma,\mu,-\lambda+\mu)$

$\Rightarrow \lambda+\mu+\gamma = a,\ \mu= b,\ -\lambda+\mu= c$

$\Rightarrow \lambda = b-c,\ \mu= b,\ \gamma = a-2b+c$.

Hence, the coordinates of vector $(a,b,c) \in R^3$ relative to the given ordered basis are $(b - c, b, a-2b+c)$

Q3. Find the coordinate vector of $v = (1,1,1)$ relative to the basis $B = \{v_1 = (1,2,3), v_2 = (-4,5,6), v_3 = (7,-8,9)\}$ of vector space R^3.

SOLUTION : Let $x,y,z \in R$ be such that

$v = xv_1 + yv_2 + zv_3$

$\Rightarrow (1,1,1) = x(1,2,3) + y(-4,5,6) + z(7,-8,9)$

$\Rightarrow (1,1,1) = (x-4y+7z,\ 2x+5y-8z,\ 3x+6y+9z)$

$\Rightarrow x-4y+7z = 1,\ 2x+5y-8z = 1,\ 3x+6y+9z = 1$

$\Rightarrow x = 7/10,\ y = -2/15,\ z = -1/30$

Hence $[v]_B = \begin{bmatrix} 7/10 \\ -2/15 \\ -1/30 \end{bmatrix}$ is the coordinate vector of v relative to basis B.

Linear Transformations

Let V and W be vector spaces. A function $T : V \to W$ is called a linear transformation if for any vectors u, v in V and scalar c,

(a) $T(u + v) = T(u) + T(v)$,

(b) $T(cu) = cT(u)$.

The inverse images $T^{-1}(0)$ of 0 is called the kernel of T and T(V) is called the range of T

Example

Q.1. Let A is an $m \times m$ matrix and B an $n \times n$ matrix.

The function $F: M(m, n) \to M(m, n)$, $F(X) = AXB$
is a linear transformation. For instance, for $m = n = 2$,

Solution: let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$.

Then $F : M(2, 2) \to M(2, 2)$ is given by

$$F(X) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, =$$

$$\begin{bmatrix} 2x_1 + 2x_2 + 4x_3 + 4x_4 & x_1 + 3x_2 + 2x_3 + 6x_4 \\ 2x_1 + 2x_2 + 6x_3 + 6x_4 & x_1 + 3x_2 + 3x_3 + 9x_4 \end{bmatrix}$$

Q2. Show that the mappings T : $R^2 \rightarrow R^3$ defined by T(x, y)= (x+1, 2y, x+y)

Solution: Let u = (x_1, y_1) and v= $(x_2, y_2) \in R^2$

Then u+ v = $(x_1, y_1) + (x_2, y_2)$

$=(x_1 + x_2, y_1 + y_2) \in R^2$

$T(u+ v) = T(x_1 + x_2, y_1 + y_2)$

$=(x_1 + x_2+1, 2(y_1 + y_2), (x_1 + x_2) + (y_1 + y_2))$

$= (x_1 + x_2+1, 2y_1 + 2y_2, (x_1 + y_1) + (x_2+ y_2))$...... (1)

And $T(u) +T(v) = T(x_1, y_1) + T(x_2, y_2)$

$= (x_1 + 1, 2y_1\ x_1 + y_1)+(x_2+ 1, 2y_1, x_2+ y_2)$

$= ((x_1 + 1) +(x_2+ 1, 2y_1+2\ y_{2,} (x_1 + y_1)+ (x_2+ y_2))$

$= ((x_1 + x_2+2, 2y_1 + 2y_2, (x_1 + y_1) + (x_2+ y_2))$...... (2)

From (1) & (2) $T(u+ v) \neq T(u) +T(v)$

Hence T is not a linear transformation.

Range: If V(F) and W(F) are vector spaces and T:V→W is a linear Transformation, Then the image set of V under T is R(T) or T(V) i.e. Range T ={T(v)/v∈ V}

A linear transformation (or linear mapping) is a mapping T: V → W such that, for each u, v ∈ V, and for each c ∈ F , T(u + v) = T(u) + T(v), and T(cu) = c T(u).

V is called the domain of the linear transformation T : V → W.

W is called the codomain of the linear transformation T : V → W.

The identity transformation $I_V : V \rightarrow V$ is defined by $I_V (v) = v$ for each v ∈ V. I_V is also denoted by I.

The zero transformation 0: V → W is defined by $0(v) = 0_W$ for each v ∈ V.

A linear operator is a linear transformation T: V → V.

Null Space or Kernel: : If V(F) and W(F) are vector spaces and T:V→W is a linear Transformation then the set of all those vectors in V whose image under T is zero, is called Kernal or Null space of T, which is denoted by N(T):

Null space of T= N(T)= { v∈ V; T(v)=0∈W}

Rank: If V(F) and W(F) are vector spaces and T:V→W is a linear Transformation then the dimension of range space of T is called the rank of T and is denoted by ρ(T).

Thus ρ(T)= dim(Range T)

Nullity: If V(F) and W(F) are vector spaces and $T:V\to W$ is a linear Transformation then the dimension of null spaces of T is called the nullity of T and is denoted by $v(T)$

Thus $v(T)=\dim(\text{Null space of } T)$.